CONFORMAL ANTI-INVARIANT SUBMERSIONS FROM ALMOST HERMITIAN MANIFOLDS

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Abstract

We introduce conformal anti-invariant submersions from almost Hermitian manifolds onto Riemannian manifolds. We give examples, investigate the geometry of foliations which are arisen from the definition of a conformal submersion and find necessary and sufficient conditions for a conformal anti-invariant submersion to be totally geodesic. We also check the harmonicity of such submersions and show that the total space has certain product structures. Moreover, we obtain curvature relations between the base space and the total space, and find geometric implications of these relations.

Keywords: Riemannian submersion, Anti-invariant submersion, Conformal submersion, conformal anti-invariant submersion.

1. Introduction

One of the main method to compare two manifolds and transfer certain structures from a manifold to another manifold is to define appropriate smooth maps between them. Given two manifolds, if the rank of a differential map is equal to the dimension of the source manifold, then such maps are called immersions and if the rank of a differential map is equal to the target manifold, then such maps are called submersions. Moreover, if these maps are isometry between manifolds, then the immersion is called

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isometric immersion (Riemannian submanifold) and the submersion is called Riemannian submersion. Riemannian submersions between Riemannian manifolds were studied by O'Neill [16] and Gray [9], for recent developments on the geometry of Riemannian submanifolds and Riemannian submersions, see:[3] and [7], respectively.

On the other hand, as a generalization of Riemannian submersions, horizontally conformal submersions are defined as follows [2]: Suppose that $(M, g_{_M})$ and $(B, g_{_B})$ are Riemannian manifolds and $F : M \longrightarrow B$ is a smooth submersion, then F is called a horizontally conformal submersion, if there is a positive function λ such that

$$\lambda^2 g_{_M}(X, Y) = g_{_B}(F_*X, F_*Y)$$

for every $X, Y \in \Gamma((kerF_*)^\perp)$. It is obvious that every Riemannian submersion is a particular horizontally conformal submersion with $\lambda = 1$. We note that horizontally conformal submersions are special horizontally conformal maps which were introduced independently by Fuglede [8] and Ishihara [13]. We also note that a horizontally conformal submersion $F : M \longrightarrow B$ is said to be horizontally homothetic if the gradient of its dilation λ is vertical, i.e.,

$$\mathcal{H}(grad\lambda) = 0 \tag{1.1}$$

at $p \in M$, where $\mathcal{H}$ is the projection on the horizontal space $(kerF_{*p})^\perp$. For conformal submersions, see: [2], [4], [5], [6], [7] and [11].

One can see that Riemannian submersions are very special maps comparing with conformal submersions. Although conformal maps does not preserve distance between points contrary to isometries, they preserve angles between vector fields. This property enables one to transfer certain properties of a manifold to another manifold by deforming such properties.

A submanifold of a complex manifold is a complex (invariant) submanifold if the tangent space of the submanifold at each point is invariant with respect to the almost complex structure of the Kähler manifold. Besides complex submanifolds of a complex

manifold, there is another important class of submanifolds called totally real submanifolds. A totally real submanifold of a complex manifold is a submanifold of such that the almost complex structure of ambient manifold carries the tangent space of the submanifold at each point into its normal space. Many authors have studied totally real submanifolds in various ambient manifolds and many interesting results were obtained, see ([3], page:322) for a survey on all these results..

As analogue of holomorphic submanifolds, holomorphic submersions were introduced by Watson [19] in seventies by using the notion of almost complex map. This notion has been extended to other manifolds, see[7] for holomorphic submersions and their extensions to other manifolds. The main property of such maps is that the vertical distributions and the horizontal distributions of such maps are invariant with respect to almost complex map. Therefore, the second author of the present paper considered a new submersion defined on an almost Hermitian manifold such that the vertical distribution is anti-invariant with respect to almost complex structure [18]. He showed that such submersions have rich geometric properties and they are useful for investigating the geometry of the total space. This new class of submersions which is called anti-invariant submersions can be seen as an analogue of totally real submanifolds in the submersion theory. Anti-invariant submersions have been also studied for different total manifolds, see: [1], [14] and [15].

As a generalization of holomorphic submersions, conformal holomorphic submersions were studied by Gudmundsson and Wood [12]. They obtained necessary and sufficient conditions for conformal holomorphic submersions to be a harmonic morphism, see also [4], [5] and [6] for the harmonicity of conformal holomorphic submersions.

In this paper, we study conformal anti-invariant submersions as a generalization of anti-invariant Riemannian submersions and investigate the geometry of the total space and the base space for the existence of such submersions. The paper is organized as follows. In the second section, we gather main notions and formulas for other sections.

In section 3, we introduce conformal anti-invariant submersions from almost Hermitian manifolds onto Riemannian manifolds, give examples and investigates the geometry of leaves of the horizontal distribution and the vertical distribution. In section 4, we find necessary and sufficient conditions for a conformal anti-invariant submersion to be harmonic and totally geodesic, respectively. In section 5, we show that there are certain product structures on the total space of a conformal anti-invariant submersion. In section 6, we study curvature relations between the total space and the base space, find several inequalities and obtain new results when the inequality becomes the equality.

2. Preliminaries

In this section, we define almost Hermitian manifolds, recall the notion of (*horizontally*) conformal submersions between Riemannian manifolds and give a brief review of basic facts of (*horizontally*) conformal submersions.

Let (M, g) be an almost Hermitian manifold. This means [20] that M admits a tensor field J of type $(1,1)$ on M such that, $\forall X, Y \in \Gamma(TM)$, we have

$$J^2 = -I, \; g(X,Y) = g(JX, JY). \tag{2.1}$$

An almost Hermitian manifold M is called Kähler manifold if

$$(\nabla_X J)Y = 0, \; \forall X, Y \in \Gamma(TM), \tag{2.2}$$

where ∇ is the Levi-Civita connection on M.

Conformal submersions belong to a wide class of conformal maps that we are going to recall their definition, but we will not study such maps in this paper.

Definition 2.1. *([2])Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds, and let $x \in M$. Then φ is called horizontally weakly conformal or semiconformal at x if either*

(i) $d\varphi_x = 0$, or

(ii) $d\varphi_x$ *maps the horizontal space* $\mathcal{H}_x = \{\ker(d\varphi_x)\}^{\perp}$ *conformally onto* $T_{\varphi(x)}N$, *i.e.,* $d\varphi_x$ *is surjective and there exists a number* $\Lambda(x) \neq 0$ *such that*

$$h(d\varphi_x(X), d\varphi_x(Y)) = \Lambda(x)g(X,Y) \ (X, Y \in \mathcal{H}_x). \tag{2.3}$$

Note that we can write the last equation more succinctly as

$$(\varphi^* h)_x \mid_{\mathcal{H}_x \times \mathcal{H}_x} = \Lambda(x) g_x \mid_{\mathcal{H}_x \times \mathcal{H}_x} .$$

With the above definition of critical point, a point x is of type (i) in Definition 2.1 if and only if it is a critical point of φ; we shall call a point of type (ii) a *regular point.* At a critical point, $d\varphi_x$ has rank 0; at a regular point, $d\varphi_x$ has rank n and φ is a submersion. The number $\Lambda(x)$ is called the *square dilation* (of φ at x); it is necessarily non-negative; its square root $\lambda(x) = \sqrt{\Lambda(x)}$ is called the dilation (of φ at x). The map φ is called *horizontally weakly conformal* or *semiconformal* (on M) if it is horizontally weakly conformal at every point of M. It is clear that if φ has no critical points, then we call it a (*horizontally*) conformal submersion.

Next, we recall the following definition from [11]. Let $\pi : M \to N$ be a submersion. A vector field E on M is said to be projectable if there exists a vector field $\breve{E}$ on N, such that $d\pi(E_x) = \breve{E}_{\pi(x)}$ for all $x \in M$. In this case E and $\breve{E}$ are called $\pi-$related. A horizontal vector field Y on (M, g) is called basic, if it is projectable. It is a well known fact that if $\breve{Z}$ is a vector field on N, then there exists a unique basic vector field Z on M, such that Z and $\breve{Z}$ are $\pi-$related. The vector field Z is called the horizontal lift of $\breve{Z}$.

The fundamental tensors of a submersion were introduced in [16]. They play a similar role to that of the second fundamental form of an immersion. More precisely, O'Neill's tensors T and A defined for vector fields E, F on M by

$$A_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F \tag{2.4}$$

$$T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F \tag{2.5}$$

where $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections (see [7]). On the other hand, from (2.4) and (2.5), we have

$$\nabla_V W = T_V W + \hat{\nabla}_V W \tag{2.6}$$

$$\nabla_V X = \mathcal{H}\nabla_V X + T_V X \tag{2.7}$$

$$\nabla_X V = A_X V + \mathcal{V}\nabla_X V \tag{2.8}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + A_X Y \tag{2.9}$$

for $X, Y \in \Gamma((\ker \pi_*)^\perp)$ and $V, W \in \Gamma(\ker \pi_*)$, where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$. If X is basic, then $\mathcal{H}\nabla_V X = A_X V$.

It is easily seen that for $x \in M$, $X \in \mathcal{H}_x$ and $V \in \mathcal{V}_x$ the linear operators $T_V, A_X : T_x M \to T_x M$ are skew-symmetric, that is

$$-g(T_V E, F) = g(E, T_V F) \text{ and } -g(A_X E, F) = g(E, A_X F)$$

for all $E, F \in T_x M$. We also see that the restriction of T to the vertical distribution $T\mid_{\mathcal{V}\times\mathcal{V}}$ is exactly the second fundamental form of the fibres of π. Since T_V is skew-symmetric we get: π has totally geodesic fibres if and only if $T \equiv 0$. For the special case when π is horizontally conformal we have the following:

Proposition 2.2. *([11]) Let $\pi : (M^m, g) \to (N^n, h)$ be a horizontally conformal submersion with dilation λ and X, Y be horizontal vectors, then*

$$A_X Y = \frac{1}{2}\{\mathcal{V}[X, Y] - \lambda^2 g(X, Y)\,\mathrm{grad}_{\mathcal{V}}(\frac{1}{\lambda^2})\}. \tag{2.10}$$

We see that the skew-symmetric part of $A\mid_{\mathcal{H}\times\mathcal{H}}$ measures the obstruction integrability of the horizontal distribution $\mathcal{H}$.

We now recall the following curvature relations for a conformal submersion from [10] and [11].

Theorem 2.3. *Let $m > n \geq 2$ and (M^m, g, ∇, R), (N^n, h, ∇^N, R^N) be two Riemannian manifolds with their Levi-Civita connections and the corresponding curvature tensors. Let $\pi : (M, g) \to (N, h)$ be a horizontally conformal submersion, with dilation $\lambda : M \to \mathbb{R}^+$ and let $R^{\mathcal{V}}$ be the curvature tensor of the fibres of π. If X, Y, Z, H are horizontal and U, V, W, F vertical vectors, then*

$$g(R(U,V)W,F) = g(R^{\mathcal{V}}(U,V)W,F) + g(T_UW, T_VF) - g(T_VW, T_UF), \tag{2.11}$$

$$g(R(U,V)W,X) = g((\nabla_U T)_V W, X) - g((\nabla_V T)_U W, X), \tag{2.12}$$

$$\begin{aligned} g(R(U,X)Y,V) &= g((\nabla_U A)_X Y, V) + g(A_X U, A_Y V) \\ &\quad - g((\nabla_X T)_U Y, V) - g(T_V Y, T_U X) \\ &\quad + \lambda^2 g(A_X Y, U) g(V, \mathrm{grad}_{\mathcal{V}}(\frac{1}{\lambda^2})), \end{aligned} \tag{2.13}$$

$$\begin{aligned} g(R(X,Y)Z,H) &= \frac{1}{\lambda^2} h(R^N(\check{X},\check{Y})\check{Z},\check{H}) + \frac{1}{4}[g(\mathcal{V}[X,Z],\mathcal{V}[Y,H]) \\ &\quad - g(\mathcal{V}[Y,Z],\mathcal{V}[X,H]) + 2g(\mathcal{V}[X,Y],\mathcal{V}[Z,H])] \\ &\quad + \frac{\lambda^2}{2}[g(X,Z)g(\nabla_Y \mathrm{grad}(\frac{1}{\lambda^2}),H) - g(Y,Z)g(\nabla_X \mathrm{grad}(\frac{1}{\lambda^2}),H) \\ &\quad + g(Y,H)g(\nabla_X \mathrm{grad}(\frac{1}{\lambda^2}),Z) - g(X,H)g(\nabla_Y \mathrm{grad}(\frac{1}{\lambda^2}),Z)] \\ &\quad + \frac{\lambda^4}{4}[(g(X,H)g(Y,Z) - g(Y,H)g(X,Z)) \parallel \mathrm{grad}(\frac{1}{\lambda^2}) \parallel^2 \\ &\quad + g(X(\frac{1}{\lambda^2})Y - Y(\frac{1}{\lambda^2})X, H(\frac{1}{\lambda^2})Z - Z(\frac{1}{\lambda^2})H)]. \end{aligned} \tag{2.14}$$

We also recall the notion of harmonic maps between Riemannian manifolds. Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $\varphi : M \to N$ is a smooth map between them. Then the differential of φ_* of φ can be viewed a section of

the bundle $Hom(TM, \varphi^{-1}TN) \to M$, where $\varphi^{-1}TN$ is the pullback bundle which has fibres $(\varphi^{-1}TN)_p = T_{\varphi(p)}N$, $p \in M$. $Hom(TM, \varphi^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. Then the second fundamental form of φ is given by

$$(\nabla\varphi_*)(X, Y) = \nabla^{\varphi}_X \varphi_*(Y) - \varphi_*(\nabla^M_X Y) \tag{2.15}$$

for $X, Y \in \Gamma(TM)$, where ∇^{φ} is the pullback connection. It is known that the second fundamental form is symmetric. A smooth map $\varphi : (M, g_M) \to (N, g_N)$ is said to be harmonic if $trace(\nabla\varphi_*) = 0$. On the other hand, the tension field of φ is the section $\tau(\varphi)$ of $\Gamma(\varphi^{-1}TN)$ defined by

$$\tau(\varphi) = \operatorname{div}\varphi_* = \sum_{i=1}^{m} (\nabla\varphi_*)(e_i, e_i), \tag{2.16}$$

where $\{e_1, ..., e_m\}$ is the orthonormal frame on M. Then it follows that φ is harmonic if and only if $\tau(\varphi) = 0$, for details, see [2].

Finally, we recall the following lemma from [2].

Lemma 2.4. *(Second fundamental form of an HC submersion) Suppose that $\varphi : M \to N$ is a horizontally conformal submersion. Then, for any horizontal vector fields X, Y and vertical vector fields V, W, we have*

(*i*) $\nabla d\varphi(X, Y) = X(\ln\lambda)d\varphi(Y) + Y(\ln\lambda)d\varphi(X) - g(X, Y)d\varphi(\operatorname{grad}\ln\lambda)$;

(*ii*) $\nabla d\varphi(V, W) = -d\varphi(A^{\mathcal{V}}_V W)$;

(*iii*) $\nabla d\varphi(X, V) = -d\varphi(\nabla^M_X V) = d\varphi((A^{\mathcal{H}})^*_X V)$.

Here $(A^{\mathcal{H}})^*_X$ is the adjoint of $A^{\mathcal{H}}_X$ characterized by

$$\langle (A^{\mathcal{H}})^*_X E, F\rangle = \langle E, A^{\mathcal{H}}_X F\rangle \ (E, F \in \Gamma(TM)).$$

3. **Conformal Anti-invariant Submersions**

In this section, we define conformal anti-invariant submersions from an almost Hermitian manifold onto a Riemannian manifold and investigate the effect of the existence of conformal anti-invariant submersions on the source manifold and the target manifold. But we first present the following notion.

Definition 3.1. *Let M be a complex m-dimensional almost Hermitian manifold with Hermitian metric g and almost complex structure J and N be a Riemannian manifold with Riemannian metric g'. A horizontally conformal submersion $F : (M^m, g) \to (N^n, g')$ with dilation λ is a called conformal anti-invariant submersion if the distribution $\ker F_*$ is anti-invariant with respect to J, i.e., $J(\ker F_*) \subseteq (\ker F_*)^\perp$.*

Let $F : (M, g, J) \to (N, g')$ be a conformal anti-invariant submersion from an almost Hermitian manifold (M, g, J) to a Riemannian manifold (N, g'). First of all, from Definition 3.1, we have $J(\ker F_*)^\perp \cap \ker F_* \neq \{0\}$. We denote the complementary orthogonal distribution to $J(\ker F_*)$ in $(\ker F_*)^\perp$ by μ. Then we have

$$(\ker F_*)^\perp = J(\ker F_*) \oplus \mu. \tag{3.1}$$

It is easy to see that μ is an invariant distribution of $(\ker F_*)^\perp$, under the endomorphism J. Thus, for $X \in \Gamma((\ker F_*)^\perp)$, we have

$$JX = BX + CX, \tag{3.2}$$

where $BX \in \Gamma(\ker F_*)$ and $CX \in \Gamma(\mu)$. On the other hand, since $F_*((\ker F_*)^\perp) = TN$ and F is a conformal submersion, using (3.2) we derive $\frac{1}{\lambda^2} g'(F_* JV, F_* CX) = 0$, for every $X \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$, which implies that

$$TN = F_*(J(\ker F_*)) \oplus F_*(\mu). \tag{3.3}$$

Example 3.2. *Every anti-invariant Riemannian submersion is a conformal anti-invariant submersion with $\lambda = I$, where I is the identity function.*

We say that a conformal anti-invariant submersion is proper if $\lambda \neq I$. We now present an example of a proper conformal anti-invariant submersion. In the following R^{2m} denotes the Euclidean $2m$-space with the standard metric. An almost complex structure J on R^{2m} is said to be compatible if (R^{2m}, J) is complex analytically isometric to the complex number space C^m with the standard flat Kählerian metric. We denote by J the compatible almost complex structure on R^{2m} defined by

$$J(a^1, ..., a^{2m}) = (-a^2, a^1, ..., -a^{2m}, a^{2m-1}).$$

Example 3.3. *Let F be a map defined by*

$$\begin{array}{cccc} F: & R^4 & \longrightarrow & R^2 \\ & (x_1, x_2, x_3, x_4) & & (e^{x_3}\sin x_4, e^{x_3}\cos x_4). \end{array}$$

Then F is a conformal anti-invariant submersion with $\lambda = e^{x_3}$.

Lemma 3.4. *Let F be a conformal anti-invariant submersion from a Kähler manifold (M, g, J) to a Riemannian manifold (N, g'). Then we have*

$$g(CY, JV) = 0 \tag{3.4}$$

and

$$g(\nabla_X CY, JV) = -g(CY, JA_X V) \tag{3.5}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$.

Proof. For $Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$, since $BY \in \Gamma(\ker F_*)$ and $JV \in \Gamma((\ker F_*)^\perp)$, using (2.1), we get (3.4). Now, using (3.4), (2.2) and (2.8) we obtain

$$g(\nabla_X CY, JV) = -g(CY, JA_X V) - g(CY, J\mathcal{V}\nabla_X V).$$

Since $J\mathcal{V}\nabla_X V \in \Gamma(J\ker F_*)$, we obtain (3.5). $\square$

We now study the integrability of the distribution $(\ker F_*)^\perp$ and then we investigate the geometry of leaves of $\ker F_*$ and $(\ker F_*)^\perp$. We note that it is known that the distribution $\ker F_*$ is integrable.

Theorem 3.5. *Let F be a conformal anti-invariant submersion from a Kähler manifold (M, g, J) to a Riemannian manifold (N, g'). Then the following assertions are equivalent to each other;*

a) $(\ker F_*)^\perp$ *is integrable,*

b)
$$\begin{aligned}\frac{1}{\lambda^2}g'(\nabla^F_Y F_*CX - \nabla^F_X F_*CY, F_*JV) = \; & g(A_X BY - A_Y BX, JV)\\ & - g(\mathcal{H}\operatorname{grad}\ln\lambda, CY)g(X, JV)\\ & + g(\mathcal{H}\operatorname{grad}\ln\lambda, CX)g(Y, JV)\\ & - 2g(CX, Y)g(\mathcal{H}\operatorname{grad}\ln\lambda, JV)\end{aligned}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$.

Proof. For $Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$, we see from Definition 3.1, $JV \in \Gamma((\ker F_*)^\perp)$ and $JY \in \Gamma(\ker F_* \oplus \mu)$. Thus using (2.1) and (2.2), for $X \in \Gamma((\ker F_*)^\perp)$ we get

$$g([X, Y], V) = g(\nabla_X JY, JV) - g(\nabla_Y JX, JV).$$

Then from (3.2) we have

$$\begin{aligned} g([X, Y], V) &= g(\nabla_X BY, JV) + g(\nabla_X CY, JV)\\ &\quad - g(\nabla_Y BX, JV) - g(\nabla_Y CX, JV).\end{aligned}$$

Since F is a conformal submersion, using (2.8) and (2.9) we arrive at

$$g([X, Y], V) = g(A_X BY - A_Y BX, JV) + \frac{1}{\lambda^2}g'(F_*\nabla_X CY, F_*JV) - \frac{1}{\lambda^2}g'(F_*\nabla_Y CX, F_*JV).$$

Thus, from (2.15) and Lemma 2.4 (i) we derive

$$\begin{aligned} g([X,Y],V) &= g(A_XBY - A_YBX, JV) - g(\mathcal{H}\operatorname{grad}\ln\lambda, X)g(CY, JV) \\ &\quad - g(\mathcal{H}\operatorname{grad}\ln\lambda, CY)g(X, JV) + g(X, CY)g(\mathcal{H}\operatorname{grad}\ln\lambda, JV) \\ &\quad + \frac{1}{\lambda^2}g'(\nabla_{F_*X}F_*CY, F_*JV) + g(\mathcal{H}\operatorname{grad}\ln\lambda, Y)g(CX, JV) \\ &\quad + g(\mathcal{H}\operatorname{grad}\ln\lambda, CX)g(Y, JV) - g(Y, CX)g(\mathcal{H}\operatorname{grad}\ln\lambda, JV) \\ &\quad - \frac{1}{\lambda^2}g'(\nabla_{F_*Y}F_*CX, F_*JV). \end{aligned}$$

Moreover, using (3.4), we obtain

$$\begin{aligned} g([X,Y],V) &= g(A_XBY - A_YBX, JV) - g(\mathcal{H}\operatorname{grad}\ln\lambda, CY)g(X, JV) \\ &\quad + g(\mathcal{H}\operatorname{grad}\ln\lambda, CX)g(Y, JV) - 2g(CX, Y)g(\mathcal{H}\operatorname{grad}\ln\lambda, JV) \\ &\quad - \frac{1}{\lambda^2}g'(\nabla_{F_*Y}F_*CX - \nabla_{F_*X}F_*CY, F_*JV), \end{aligned}$$

which proves $(a) \Leftrightarrow (b)$. □

From Theorem 3.5, we deduce the following which shows that a conformal anti-invariant submersion with integrable $(\ker F_*)^\perp$ turns out to be a horizontally homothetic submersion.

Theorem 3.6. *Let F be a conformal anti-invariant submersion from a Kähler manifold (M, g, J) to a Riemannian manifold (N, g'). Then any two conditions below imply the three:*

(i) $(\ker F_*)^\perp$ *is integrable*

(ii) F *is horizontally homotetic.*

(iii) $g'(\nabla^F_Y F_*CX - \nabla^F_X F_*CY, F_*JV) = \lambda^2 g(A_XBY - A_YBX, JV)$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$.

Proof. For $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$, from Theorem 3.5, we have

$$g([X,Y],V) = g(A_X BY - A_Y BX, JV) - g(\mathcal{H}\,\mathrm{grad}\ln\lambda, CY)g(X,JV)$$
$$+ g(\mathcal{H}\,\mathrm{grad}\ln\lambda, CX)g(Y,JV) - 2g(CX,Y)g(\mathcal{H}\,\mathrm{grad}\ln\lambda, JV)$$
$$- \frac{1}{\lambda^2} g'(\nabla_{F_*Y} F_*CX - \nabla_{F_*X} F_*CY, F_*JV).$$

Now, if we have (i) and (iii), then we arrive at

$$-g(\mathcal{H}\,\mathrm{grad}\ln\lambda, CY)g(X,JV) + g(\mathcal{H}\,\mathrm{grad}\ln\lambda, CX)g(Y,JV)$$
$$-2g(CX,Y)g(\mathcal{H}\,\mathrm{grad}\ln\lambda, JV) = 0. \tag{3.6}$$

Now, taking $Y = JV$ in (3.6) for $V \in \Gamma(ker F_*)$ and using (3.4), we get

$$g(\mathcal{H}\,\mathrm{grad}\ln\lambda, CX)g(V,V) = 0.$$

Hence λ is a constant on $\Gamma(\mu)$. On the other hand, taking $Y = CX$ in (3.6) for $X \in \Gamma(\mu)$ and using (3.4) we derive

$$-g(\mathcal{H}\,\mathrm{grad}\ln\lambda, C^2X)g(X,JV) + g(\mathcal{H}\,\mathrm{grad}\ln\lambda, CX)g(CX,JV)$$
$$- 2g(CX,CX)g(\mathcal{H}\,\mathrm{grad}\ln\lambda, JV) = 0,$$

hence, we arrive at

$$g(CX,CX)g(\mathcal{H}\,\mathrm{grad}\ln\lambda, JV) = 0.$$

From above equation, λ is a constant on $\Gamma(J(\ker F_*))$. Similarly, one can obtain the other assertions. □

We say that a conformal anti-invariant submersion is a conformal Lagrangian submersion if $J(\ker F_*) = (\ker F_*)^\perp$. From Theorem 3.5, we have the following.

Corollary 3.7. *Let $F:(M,g,J)\to(N,g')$ be a conformal Lagrangian submersion, where (M,g,J) is a Kähler manifold and (N,g') is a Riemannian manifold. Then the following assertions are equivalent to each other;*

(*i*) $(\ker F_*)^\perp$ *is integrable.*

(*ii*) $A_X JY = A_Y JX$

(*iii*) $(\nabla F_*)(Y,JX) = (\nabla F_*)(X,JY)$

for $X,Y\in\Gamma((\ker F_*)^\perp)$.

Proof. For $X,Y\in\Gamma((\ker F_*)^\perp)$ and $V\in\Gamma(\ker F_*)$, we see from Definition 3.1, $JV\in\Gamma((\ker F_*)^\perp)$ and $JY\in\Gamma(J(\ker F_*))$. From Theorem 3.5 we have

$$\begin{aligned} g([X,Y],V) = g(A_XBY - A_YBX, JV) &- g(\mathcal{H}\operatorname{grad}\ln\lambda, CY)g(X,JV)\\ &+ g(\mathcal{H}\operatorname{grad}\ln\lambda, CX)g(Y,JV) - 2g(CX,Y)g(\mathcal{H}\operatorname{grad}\ln\lambda, JV)\\ &- \frac{1}{\lambda^2}g'(\nabla_{F_*Y}F_*CX - \nabla_{F_*X}F_*CY, F_*JV). \end{aligned}$$

Since F is a conformal Lagrangian submersion, we derive

$$g([X,Y],V) = g(A_XBY - A_YBX, JV) = 0$$

which shows $(i)\Leftrightarrow(ii)$. On the other hand using Definition 3.1 and (2.8) we arrive at

$$\begin{aligned} g(A_XBY,JV) - g(A_YBX,JV) &= \frac{1}{\lambda^2}g'(F_*A_XBY, F_*JV) - \frac{1}{\lambda^2}g'(F_*A_YBX, F_*JV)\\ &= \frac{1}{\lambda^2}g'(F_*(\nabla_XBY), F_*JV) - \frac{1}{\lambda^2}g'(F_*(\nabla_YBX), F_*JV). \end{aligned}$$

Now, using (2.15) we obtain

$$\begin{aligned} &\frac{1}{\lambda^2}\{g'(F_*(\nabla_XBY), F_*JV) - g'(F_*(\nabla_YBX), F_*JV)\}\\ &\qquad = \frac{1}{\lambda^2}g'(-(\nabla F_*)(X,BY) + \nabla_{F_*X}F_*BY, F_*JV)\\ &\qquad - \frac{1}{\lambda^2}g'(-(\nabla F_*)(Y,BX) + \nabla_{F_*Y}F_*BX, F_*JV) \end{aligned}$$

$$= \frac{1}{\lambda^2}\{g'((\nabla F_*)(Y, BX) - (\nabla F_*)(X, BY), F_*JV)$$

which tells that $(ii) \Leftrightarrow (iii)$. □

For the geometry of leaves of the horizontal distribution, we have the following theorem.

Theorem 3.8. *Let F be a conformal anti-invariant submersion from a Kähler manifold (M, g, J) to a Riemannian manifold (N, g'). Then the following assertions are equivalent to each other;*

(i) $(\ker F_*)^\perp$ *defines a totally geodesic foliation on M.*

(ii) $\frac{1}{\lambda^2} g'(\nabla_{F_*X} F_*CY, F_*JV) = -g(A_X BY, JV) + g(\mathcal{H}\,\mathrm{grad}\ln\lambda, CY)g(X, JV) - g(\mathcal{H}\,\mathrm{grad}\ln\lambda, JV)g(X, CY)$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$.

Proof. From (2.1), (2.2), (2.8), (2.9), (3.2) and (3.1) we get

$$g(\nabla_X Y, V) = g(A_X BY, JV) + g(\mathcal{H}\nabla_X CY, JV).$$

Since F is a conformal submersion, using (2.15) and Lemma 2.4 (i) we arrive at

$$\begin{aligned} g(\nabla_X Y, V) = {} & g(A_X BY, JV) - \frac{1}{\lambda^2} g(\mathcal{H}\,\mathrm{grad}\ln\lambda, X) g'(F_*CY, F_*JV) \\ & - \frac{1}{\lambda^2} g(\mathcal{H}\,\mathrm{grad}\ln\lambda, CY) g'(F_*X, F_*JV) + \frac{1}{\lambda^2} g(X, CY) g'(F_*(\mathrm{grad}\ln\lambda), F_*JV) \\ & + \frac{1}{\lambda^2} g'(\nabla_{F_*X} F_*CY, F_*JV). \end{aligned}$$

Moreover, using Definition 3.1 and (3.4) we obtain

$$\begin{aligned} g(\nabla_X Y, V) = {} & g(A_X BY, JV) - g(\mathcal{H}\,\mathrm{grad}\ln\lambda, CY) g(X, JV) \\ & + g(\mathcal{H}\,\mathrm{grad}\ln\lambda, JV) g(X, CY) + \frac{1}{\lambda^2} g'(\nabla_{F_*X} F_*CY, F_*JV) \end{aligned}$$

which proves $(i) \Leftrightarrow (ii)$. □

From Theorem 3.8, we also deduce the following characterization.

Theorem 3.9. *Let F be a conformal anti-invariant submersion from a Kähler manifold (M,g,J) to a Riemannian manifold (N,g'). Then any two conditions below imply the three:*

(i) $(\ker F_*)^\perp$ *defines a totally geodesic foliation on M.*

(ii) *F is horizontally homotetic.*

(iii) $g'(\nabla_{F_*X} F_*CY, F_*JV) = -\lambda^2 g(A_X BY, JV)$

for $X,Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$.

Proof. For $X,Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$, from Theorem 3.8, we have

$$g(\nabla_X Y, V) = g(A_X BY, JV) - g(\mathcal{H}\,\mathrm{grad}\ln\lambda, CY)g(X,JV) + g(\mathcal{H}\,\mathrm{grad}\ln\lambda, JV)g(X,CY) + \frac{1}{\lambda^2}g'(\nabla_{F_*X} F_*CY, F_*JV).$$

Now, if we have (i) and (iii), then we obtain

$$-g(\mathcal{H}\,\mathrm{grad}\ln\lambda, CY)g(X,JV) + g(\mathcal{H}\,\mathrm{grad}\ln\lambda, JV)g(X,CY) = 0. \tag{3.7}$$

Now, taking $X = CY$ in (3.7) and using (3.4), we get

$$g(\mathcal{H}\,\mathrm{grad}\ln\lambda, JV)g(CY,CY) = 0.$$

Thus, λ is a constant on $\Gamma(J(\ker F_*))$. On the other hand, taking $X = JV$ in (3.7) and using (3.4) we derive

$$g(\mathcal{H}\,\mathrm{grad}\ln\lambda, CY)g(V,V) = 0.$$

From above equation, λ is a constant on $\Gamma(\mu)$. Similarly, one can obtain the other assertions. □

In particular, if F is a conformal Lagrangian submersion, then we have the following.

Corollary 3.10. *Let $F : (M, g, J) \to (N, g')$ be a conformal Lagrangian submersion, where (M, g, J) is a Kähler manifold and (N, g') is a Riemannian manifold. Then the following assertions are equivalent to each other;*

(i) $(\ker F_*)^\perp$ *defines a totally geodesic foliation on M.*

(ii) $A_X JY = 0$

(iii) $(\nabla F_*)(X, JY) = 0$

for $X, Y \in \Gamma((\ker F_*)^\perp)$.

Proof. For $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$, we see from Definition 3.1, $JV \in \Gamma((\ker F_*)^\perp)$ and $JY \in \Gamma(J(\ker F_*))$. From Theorem 3.8 we have

$$g(\nabla_X Y, V) = g(A_X BY, JV) - g(\mathcal{H}\,\mathrm{grad}\ln\lambda, CY)g(X, JV) + g(\mathcal{H}\,\mathrm{grad}\ln\lambda, JV)g(X, CY) + \frac{1}{\lambda^2}g'(\nabla_{F_*X} F_*CY, F_*JV).$$

Since F is a conformal Lagrangian submersion, we derive

$$g(\nabla_X Y, V) = g(A_X BY, JV)$$

which shows $(i) \Leftrightarrow (ii)$. On the other hand using (2.8) we get

$$g(A_X BY, JV) = g(\nabla_X BY, JV).$$

Since F is a conformal submersion, we have

$$g(A_X BY, JV) = \frac{1}{\lambda^2}g'(F_*\nabla_X BY, F_*JV).$$

Then using (2.15) we get

$$g(A_X BY, JV) = -\frac{1}{\lambda^2}g'((\nabla F_*)(X, BY), F_*JV)$$

which tells that $(ii) \Rightarrow (iii)$. □

In the sequel we are going to investigate the geometry of leaves of the distribution $\ker F_*$.

Theorem 3.11. *Let $F : (M, g, J) \to (N, g')$ be a conformal anti-invariant submersion, where (M, g, J) is a Kähler manifold and (N, g') is a Riemannian manifold. Then the following assertions are equivalent to each other;*

(i) $\ker F_*$ defines a totally geodesic foliation on M.

(ii) $-\frac{1}{\lambda^2} g'(\nabla_{F_* JW} F_* JV, F_* JCX) = g(T_V JW, BX) + g(JW, JV) g(\mathcal{H} \operatorname{grad} \ln \lambda, JCX)$ for $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.

Proof. For $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$, from (2.1), (2.2), (2.7) and (3.2) we get

$$g(\nabla_V W, X) = g(T_V JW, BX) + g(\mathcal{H} \nabla_V JW, CX).$$

Since ∇ is torsion free and $[V, JW] \in \Gamma(\ker F_*)$, we obtain

$$g(\nabla_V W, X) = g(T_V JW, BX) + g(\nabla_{JW} V, CX).$$

Using (2.2) and (2.9) we have

$$g(\nabla_V W, X) = g(T_V JW, BX) + g(\nabla_{JW} JV, JCX),$$

here we have used that μ is invariant. Since F is a conformal submersion, using (2.15) and Lemma 2.4 (i) we obtain

$$\begin{aligned} g(\nabla_V W, X) = {} & g(T_V JW, BX) - \frac{1}{\lambda^2} g(\mathcal{H} \operatorname{grad} \ln \lambda, JW) g'(F_* JV, F_* JCX) \\ & - \frac{1}{\lambda^2} g(\mathcal{H} \operatorname{grad} \ln \lambda, JV) g'(F_* JW, F_* JCX) + g(JW, JV) \frac{1}{\lambda^2} g'(F_* \operatorname{grad} \ln \lambda, F_* JCX) \\ & + \frac{1}{\lambda^2} g'(\nabla_{F_* JW} F_* JV, F_* JCX). \end{aligned}$$

Moreover, using Definition 3.1 and (3.4) we derive

$$g(\nabla_V W, X) = g(T_V JW, BX) + g(JW, JV) g(\mathcal{H}\,\mathrm{grad}\ln\lambda, JCX) + \frac{1}{\lambda^2} g'(\nabla_{F_* JW} F_* JV, F_* JCX)$$

which proves $(i) \Leftrightarrow (ii)$. □

From Theorem 3.11, we deduce have the following result.

Theorem 3.12. *Let F be a conformal anti-invariant submersion from a Kähler manifold (M, g, J) to a Riemannian manifold (N, g'). Then any two conditions below imply the three:*

(i) $\ker F_*$ *defines a totally geodesic foliation on M.*

(ii) *λ is a constant on $\Gamma(\mu)$.*

(iii) $\frac{1}{\lambda^2} g'(\nabla_{F_* JW} F_* JV, F_* JCX) = -g(T_V JW, JX)$

for $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.

Proof. For $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$, from Theorem 3.11, we have

$$g(\nabla_V W, X) = g(T_V JW, BX) + g(JW, JV) g(\mathcal{H}\,\mathrm{grad}\ln\lambda, JCX) + \frac{1}{\lambda^2} g'(\nabla_{F_* JW} F_* JV, F_* JCX).$$

Now, if we have (i) and (iii), then we get

$$g(JW, JV) g(\mathcal{H}\,\mathrm{grad}\ln\lambda, JCX) = 0.$$

From above equation, λ is a constant on $\Gamma(\mu)$. Similarly, one can obtain the other assertions. □

If F is a conformal Lagrangian submersion, then (3.3) implies that $TN = F_*(J(\ker F_*))$. Hence we have the following.

Corollary 3.13. *Let $F:(M,g,J)\rightarrow(N,g')$ be a conformal Lagrangian submersion, where (M,g,J) is a Kähler manifold and (N,g') is a Riemannian manifold. Then the following assertions are equivalent to each other;*

(i) $\ker F_*$ *defines a totally geodesic foliation on* M.

(ii) $T_V JW=0$

for $V,W\in\Gamma(\ker F_*)$.

Proof. For $V,W\in\Gamma(\ker F_*)$ and $X\in\Gamma((\ker F_*)^\perp)$, from Theorem 3.11 we have

$$g(\nabla_V W,X)=g(T_V JW,BX)+g(JW,JV)g(\mathcal{H}\operatorname{grad}\ln\lambda,JCX)+\frac{1}{\lambda^2}g'(\nabla_{F_*JW}F_*JV,F_*JCX).$$

Since F is a conformal Lagrangian submersion, we get

$$g(\nabla_V W,X)=g(T_V JW,BX)$$

which shows $(i)\Leftrightarrow(ii)$. □

4. Harmonicity of Conformal Anti-invariant Submersions

In this section, we are going to find necessary and sufficient conditions for a conformal anti-invariant submersions to be harmonic. We also investigate the necessary and sufficient conditions for such submersions to be totally geodesic.

Theorem 4.1. *Let $F:(M^{2m+2r},g,J)\rightarrow(N^{m+2r},g')$ be a conformal anti-invariant submersion, where (M,g,J) is a Kähler manifold and (N,g') is a Riemannian manifold. Then the tension field τ of F is*

$$\begin{aligned}\tau(F)= & -\tfrac{1}{m}F_*(\mu^{\ker F_*})+(\tfrac{2}{\lambda^2}-(m+2r))F_*(\operatorname{grad}\ln\lambda)\mid_{F_*(JV)}\\ & +(\tfrac{2}{\lambda^2}-(m+2r))F_*(\operatorname{grad}\ln\lambda)\mid_{F_*(\mu)}\end{aligned}\tag{4.1}$$

where $\mu^{\ker F_*}$ is the mean curvature vector field of the distribution of $\ker F_*$.

Proof. Let $\{e_1, ..., e_m, Je_1, ..., Je_m, \mu_1, ..., \mu_r, J\mu_r, ..., J\mu_r\}$ be an orthonormal basis of $\Gamma(TM)$ such that $\{e_1, ..., e_m\}$ is orthonormal basis of $\Gamma(\ker F_*)$, $\{Je_1, ..., Je_m\}$ is orthonormal basis of $\Gamma(J\ker F_*)$ and $\{\mu_1, ..., \mu_r, J\mu_r, ..., J\mu_r\}$ is orthonormal basis of $\Gamma(\mu)$. Then the trace of second fundamental form (restriction to $\ker F_* \times \ker F_*$) is given by

$$trace^{\ker F_*}\nabla F_* = \sum_{i=1}^{m}(\nabla F_*)(e_i, e_i).$$

Then using (2.15) we obtain

$$trace^{\ker F_*}\nabla F_* = -\frac{1}{m}F_*(\mu^{\ker F_*}). \tag{4.2}$$

In a similar way, we have

$$trace^{(\ker F_*)^{\perp}}\nabla F_* = \sum_{i=1}^{m}(\nabla F_*)(Je_i, Je_i) + \sum_{i=1}^{2r}(\nabla F_*)(\mu_i, \mu_i).$$

Using Lemma 2.4 (i) we arrive at

$$\begin{aligned} trace^{(\ker F_*)^{\perp}}\nabla F_* = & \sum_{i=1}^{m} 2g(\operatorname{grad}\ln\lambda, Je_i)F_*(Je_i) - mF_*(\operatorname{grad}\ln\lambda) \\ & + \sum_{i=1}^{2r} 2g(\operatorname{grad}\ln\lambda, \mu_i)F_*(\mu_i) - 2rF_*(\operatorname{grad}\ln\lambda). \end{aligned}$$

Since F is a conformal anti-invariant submersion, we derive

$$\begin{aligned} trace^{(\ker F_*)^{\perp}}\nabla F_* = & \sum_{i=1}^{m} 2\frac{1}{\lambda^2}g'(F_*(\operatorname{grad}\ln\lambda), F_*(Je_i))F_*(Je_i) - mF_*(\operatorname{grad}\ln\lambda) \\ & + \sum_{i=1}^{2r} 2\frac{1}{\lambda^2}g'(F_*(\operatorname{grad}\ln\lambda), F_*(\mu_i))F_*(\mu_i) - 2rF_*(\operatorname{grad}\ln\lambda) \qquad (4.3) \\ & = (\frac{2}{\lambda^2} - (m+2r))F_*(\operatorname{grad}\ln\lambda) \mid_{F_*(JV)} + (\frac{2}{\lambda^2} \\ & - (m+2r))F_*(\operatorname{grad}\ln\lambda) \mid_{F_*(\mu)} . \end{aligned}$$

Then proof follows from (4.2) and (4.3). □

From Theorem 4.1 we deduce that:

Theorem 4.2. *Let $F : (M^{2m+2r}, g, J) \to (N^{m+2r}, g')$ be a conformal anti-invariant submersion such that $\frac{2}{(m+2r)} \neq \lambda^2$, where (M, g, J) is a Kähler manifold and (N, g') is a Riemannian manifold. Then any three conditions below imply the fourth:*

(i) *F is harmonic*

(ii) *The fibres are minimal*

(iii) *λ is a constant on $\Gamma(J\ker F_*)$*

(iv) *λ is a constant on $\Gamma(\mu)$.*

Proof. From (4.1), we have

$$\begin{aligned}\tau(F) = \quad & -\tfrac{1}{m}F_*(\mu^{\ker F_*}) + (\tfrac{2}{\lambda^2} - (m+2r))F_*(\operatorname{grad}\ln\lambda)\mid_{F_*(JV)} \\ & +(\tfrac{2}{\lambda^2} - (m+2r))F_*(\operatorname{grad}\ln\lambda)\mid_{F_*(\mu)} .\end{aligned}$$

Now, if we have $(i), (ii)$ and (iii) then λ is a constant on $\Gamma(\mu)$. □

We also have the following result.

Corollary 4.3. *Let F be a conformal anti-invariant submersion from a Kähler manifold (M, g, J) to a Riemannian manifold (N, g'). If $\frac{2}{(m+2r)} = \lambda^2$ then F is harmonic if and only if the fibres are minimal.*

Now we obtain necessary and sufficient condition for conformal anti-invariant submersion to be totally geodesic. We recall that a differentiable map F between two Riemannian manifolds is called totally geodesic if

$$(\nabla F_*)(X, Y) = 0, \text{ for all } X, Y \in \Gamma(TM).$$

A geometric interpretation of a totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths.

Theorem 4.4. *Let F be a conformal anti-invariant submersion from a Kähler manifold (M, g, J) to a Riemannian manifold (N, g'). Then F is a totally geodesic map if and*

only if

$$-\nabla^F_X F_*Y = F_*(J(A_X JY_1 + \mathcal{V}\nabla_X BY_2 + A_X CY_2) + C(\mathcal{H}\nabla_X JY_1 + A_X BY_2 + \mathcal{H}\nabla_X CY_2)) \tag{4.4}$$

for any $X, Y = Y_1 + Y_2 \in \Gamma(TM)$, *where* $Y_1 \in \Gamma(\ker F_*)$ *and* $Y_2 \in \Gamma((\ker F_*)^\perp)$.

Proof. Using (2.2) and (2.15) we have

$$(\nabla F_*)(X, Y) = \nabla^F_X F_*Y + F_*(J\nabla_X JY)$$

for any $X, Y \in \Gamma(TM)$. Then from (2.8) and (3.2) we get

$$(\nabla F_*)(X, Y) = \nabla^F_X F_*Y + F_*(JA_X JY_1 + B\mathcal{H}\nabla_X JY_1 + C\mathcal{H}\nabla_X JY_1 + BA_X BY_2 + CA_X BY_2 + J\mathcal{V}\nabla_X BY_2 + JA_X CY_2 + B\mathcal{H}\nabla_X CY_2 + C\mathcal{H}\nabla_X CY_2)$$

for any $Y = Y_1 + Y_2 \in \Gamma(TM)$, where $Y_1 \in \Gamma(\ker F_*)$ and $Y_2 \in \Gamma((\ker F_*)^\perp)$. Thus taking into account the vertical parts, we find

$$(\nabla F_*)(X, Y) = \nabla^F_X F_*Y + F_*(J(A_X JY_1 + \mathcal{V}\nabla_X BY_2 + A_X CY_2) + C(\mathcal{H}\nabla_X JY_1 + A_X BY_2 + \mathcal{H}\nabla_X CY_2)).$$

Thus $(\nabla F_*)(X, Y) = 0$ if and only if the equation (4.4) is satisfied. □

We now present the following definition.

Definition 4.5. *Let F be a conformal anti-invariant submersion from a Kähler manifold (M, g, J) to a Riemannian manifold (N, g'). Then F is called a $(J\ker F_*, \mu)$-totally geodesic map if*

$$(\nabla F_*)(JU, X) = 0, \ \textit{for } U \in \Gamma(\ker F_*) \textit{ and } X \in \Gamma((\ker F_*)^\perp).$$

In the sequel we show that this notion has an important effect on the character of the conformal submersion.

Theorem 4.6. *Let F be a conformal anti-invariant submersion from a Kähler manifold (M, g, J) to a Riemannian manifold (N, g'). Then F is a $(J \ker F_*, \mu)$-totally geodesic map if and only if F is horizontally homotetic map.*

Proof. For $U \in \Gamma(\ker F_*)$ and $X \in \Gamma(\mu)$, from Lemma 2.4 (i), we have

$$(\nabla F_*)(JU, X) = JU(\ln \lambda)F_*(X) + X(\ln \lambda)F_*(JU) - g(JU, X)F_*(\operatorname{grad} \ln \lambda).$$

From above equation, if F is a horizontally homotetic then $(\nabla F_*)(JU, X) = 0$. Conversely, if $(\nabla F_*)(JU, X) = 0$, we obtain

$$JU(\ln \lambda)F_*(X) + X(\ln \lambda)F_*(JU) = 0. \tag{4.5}$$

Taking inner product in (4.5) with $F_*(JU)$ and since F is a conformal submersion, we write

$$g(\operatorname{grad} \ln \lambda, JU)g'(F_*X, F_*JU) + g(\operatorname{grad} \ln \lambda, X)g'(F_*JU, F_*JU) = 0.$$

Above equation implies that λ is a constant on $\Gamma(\mu)$. On the other hand, taking inner product in (4.5) with F_*X, we have

$$g(\operatorname{grad} \ln \lambda, JU)g'(F_*X, F_*X) + g(\operatorname{grad} \ln \lambda, X)g'(F_*JU, F_*X) = 0.$$

From above equation, it follows that λ is a constant on $\Gamma(J(\ker F_*))$. Thus λ is a constant on $\Gamma((\ker F_*)^{\perp})$. Hence proof is complete. $\square$

Here we present another result on conformal anti-invariant submersion to be totally geodesic.

Theorem 4.7. *Let F be a conformal anti-invariant submersion from a Kähler manifold (M, g, J) to a Riemannian manifold (N, g'). Then F is a totally geodesic map if and only if*

(i) $T_U JV = 0$ *and* $\mathcal{H}\nabla_U JV \in \Gamma(J\ker F_*)$,

(ii) *F is horizontally homotetic map,*

(iii) $\hat{\nabla}_V BX + T_V CX = 0$

$T_V BX + \mathcal{H}\nabla_V CX \in \Gamma(J\ker F_*)$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $U, V \in \Gamma(\ker F_*)$.

Proof. For any $U, V \in \Gamma(\ker F_*)$, from (2.2) and (2.15) we have

$$(\nabla F_*)(U, V) = F_*(J\nabla_U JV).$$

Then (3.2) and (2.7) implies that

$$(\nabla F_*)(U, V) = F_*(JT_U JV + C\mathcal{H}\nabla_U JV).$$

From above equation, $(\nabla F_*)(U, V) = 0$ if and only if

$$F_*(JT_U JV + C\mathcal{H}\nabla_U JV) = 0. \tag{4.6}$$

This implies $T_U JV = 0$ and $\mathcal{H}\nabla_U JV \in \Gamma(J\ker F_*)$. On the other hand, from Lemma 2.4 (i) we derive

$$(\nabla F_*)(X, Y) = X(\ln\lambda)F_*(Y) + Y(\ln\lambda)F_*(X) - g(X, Y)F_*(\operatorname{grad}\ln\lambda)$$

for any $X, Y \in \Gamma(\mu)$. It is obvious that if F is horizontally homothetic, it follows that $(\nabla F_*)(X, Y) = 0$. Conversely, if $(\nabla F_*)(X, Y) = 0$, taking $Y = JX$ in above equation, we get

$$X(\ln\lambda)F_*(JX) + JX(\ln\lambda)F_*(X) = 0. \tag{4.7}$$

Taking inner product in (4.7) with $F_* JX$, we obtain

$$g(\operatorname{grad}\ln\lambda, X)\lambda^2 g(JX, JX) + g(\operatorname{grad}\ln\lambda, JX)\lambda^2 g(X, JX) = 0. \tag{4.8}$$

From (4.8), λ is a constant on $\Gamma(\mu)$. On the other hand, for $U, V \in \Gamma(\ker F_*)$, from Lemma 2.4 (i) we have

$$(\nabla F_*)(JU, JV) = JU(\ln\lambda)F_*(JV) + JV(\ln\lambda)F_*(JU) - g(JU, JV)F_*(\operatorname{grad}\ln\lambda).$$

Again if F is horizontally homothetic, then $(\nabla F_*)(JU, JV) = 0$. Conversely, if $(\nabla F_*)(JU, JV) = 0$, putting U instead of V in above equation, we derive

$$2JU(\ln\lambda)F_*(JU) - g(JU, JU)F_*(\operatorname{grad}\ln\lambda) = 0. \tag{4.9}$$

Taking inner product in (4.9) with $F_* JU$ and since F is a conformal submersion, we have

$$g(JU, JU)\lambda^2 g(\operatorname{grad}\ln\lambda, JU) = 0.$$

From above equation, λ is a constant on $\Gamma(J\ker F_*)$. Thus λ is a constant on $\Gamma((\ker F_*)^\perp)$. Now, for $X \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$, from (2.2) and (2.15) we get

$$(\nabla F_*)(X, V) = F_*(J\nabla_V JX).$$

Using (3.2) and (2.7) we have

$$(\nabla F_*)(X, V) = F_*(CT_V BX + J\hat{\nabla}_V BX + C\mathcal{H}\nabla_V CX + JT_V CX).$$

Thus $(\nabla F_*)(X, V) = 0$ if and only if

$$F_*(CT_V BX + J\hat{\nabla}_V BX + C\mathcal{H}\nabla_V CX + JT_V CX) = 0.$$

Thus proof is complete. $\square$

5. **Decomposition Theorems**

In this section, we obtain decomposition theorems by using the existence of conformal anti-invariant submersions. First, we recall the following results from [17]. Let g be a Riemannian metric tensor on the manifold $B = M \times N$ and assume that the canonical foliations D_M and D_N intersect perpendicularly everywhere. Then g is the metric tensor of

(i) a twisted product $M \times_f N$ if and only if D_M is a totally geodesic foliation and D_N is a totally umbilic foliation,

(ii) a warped product $M \times_f N$ if and only if D_M is a totally geodesic foliation and D_N is a spheric foliation, i.e., it is umbilic and its mean curvature vector field is parallel.

(iii) a usual product of Riemannian manifolds if and only if D_M and D_N are totally geodesic foliations.

Our first decomposition theorem for a conformal anti-invariant submersion comes from Theorem 3.8 and Theorem 3.11 in terms of the second fundamental forms of such submersions.

Theorem 5.1. *Let F be a conformal anti-invariant submersion from a Kähler manifold (M, g, J) to a Riemannian manifold (N, g'). Then M is a locally product manifold if and only if*

$$\frac{1}{\lambda^2} g'(\nabla_{F_*X} F_*CY, F_*JV) = -g(A_X BY, JV) + g(\mathcal{H}\,\mathrm{grad}\ln\lambda, CY)g(X, JV) - g(\mathcal{H}\,\mathrm{grad}\ln\lambda, JV)g(X, CY)$$

and

$$-\frac{1}{\lambda^2} g'(\nabla_{F_*JW} F_*JV, F_*JCX) = g(T_V JW, BX) + g(JW, JV)g(\mathcal{H}\,\mathrm{grad}\ln\lambda, JCX)$$

for $V, W \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$.

From Corollary 3.10 and Corollary 3.13, we have the following theorem.

Theorem 5.2. *Let F be a conformal Lagrangian submersion from a Kähler manifold (M, g, J) to a Riemannian manifold (N, g'). Then M is a locally product manifold if and only if $A_X JY = 0$ and $T_V JW = 0$ for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V, W \in \Gamma(\ker F_*)$.*

Next we obtain a decomposition theorem which is related to the notion of twisted product manifold.

Theorem 5.3. *Let F be a conformal anti-invariant submersion from a Kähler manifold (M, g, J) to a Riemannian manifold (N, g'). Then M is a locally twisted product manifold of the form $M_{(\ker F_*)} \times_\lambda M_{(\ker F_*)^\perp}$ if and only if*

$$-\frac{1}{\lambda^2} g'(\nabla_{F_* JW} F_* JV, F_* JCX) = g(T_V JW, BX) + g(JW, JV) g(\mathcal{H} \operatorname{grad} \ln \lambda, JCX) \quad (5.1)$$

and

$$g(X, Y)H = -BA_X BY + CY(\ln \lambda)BX - B\mathcal{H} \operatorname{grad} \ln \lambda g(X, CY) - JF^*(\nabla_{F_* X} F_{*CY}) \quad (5.2)$$

for $V, W \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$, where $M_{(\ker F_*)^\perp}$ and $M_{(\ker F_*)}$ are integral manifolds of the distributions $(\ker F_*)^\perp$ and $(\ker F_*)$ and H is the mean curvature vector field of $M_{(\ker F_*)^\perp}$.

Proof. For $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$, from (2.1), (2.2), (2.7) and (3.2) we have

$$g(\nabla_V W, X) = g(T_V JW, BX) + g(\mathcal{H} \nabla_V JW, CX).$$

Since ∇ is torsion free and $[V, JW] \in \Gamma(\ker F_*)$, we obtain

$$g(\nabla_V W, X) = g(T_V JW, BX) + g(\nabla_{JW} V, CX).$$

Using (2.2) and (2.9) we get

$$g(\nabla_V W, X) = g(T_V JW, BX) + g(\nabla_{JW} JV, JCX).$$

Since F is a conformal submersion, using (2.15) and Lemma 2.4 (i) we arrive at

$$\begin{aligned} g(\nabla_V W, X) = g(T_V JW, BX) &- \frac{1}{\lambda^2} g(\mathcal{H}\,\mathrm{grad}\ln\lambda, JW) g'(F_* JV, F_* JCX) \\ &- \frac{1}{\lambda^2} g(\mathcal{H}\,\mathrm{grad}\ln\lambda, JV) g'(F_* JW, F_* JCX) \\ &+ g(JW, JV)\frac{1}{\lambda^2} g'(F_*\,\mathrm{grad}\ln\lambda, F_* JCX) + \frac{1}{\lambda^2} g'(\nabla_{F_* JW} F_* JV, F_* JCX). \end{aligned}$$

Moreover, using Definition 3.1 and (3.4) we conclude that

$$\begin{aligned} g(\nabla_V W, X) &= g(T_V JW, BX) + g(JW, JV) g(\mathcal{H}\,\mathrm{grad}\ln\lambda, JCX) \\ &\quad + \frac{1}{\lambda^2} g'(\nabla_{F_* JW} F_* JV, F_* JCX). \end{aligned}$$

Thus it follows that $M_{(\ker F_*)}$ is totally geodesic if and only if the equation (5.1) is satisfied. On the other hand, for $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$, from (2.1), (2.2), (2.8), (2.9) and (3.2) we obtain

$$g(\nabla_X Y, V) = g(A_X BY + \mathcal{V}\nabla_X BY, JV) + g(A_X CY + \mathcal{H}\nabla_X CY, JV).$$

Thus from (3.1) we get

$$g(\nabla_X Y, V) = g(A_X BY, JV) + g(\mathcal{H}\nabla_X CY, JV).$$

Since F is a conformal submersion, using (2.15) and Lemma 2.4 (i) we arrive at

$$\begin{aligned} g(\nabla_X Y, V) = g(A_X BY, JV) &- \frac{1}{\lambda^2} g(\mathcal{H}\,\mathrm{grad}\ln\lambda, X) g'(F_* CY, F_* JV) \\ &- \frac{1}{\lambda^2} g(\mathcal{H}\,\mathrm{grad}\ln\lambda, CY) g'(F_* X, F_* JV) + \frac{1}{\lambda^2} g(X, CY) g'(F_*(\mathrm{grad}\ln\lambda), F_* JV) \\ &+ \frac{1}{\lambda^2} g'(\nabla_{F_* X} F_* CY, F_* JV). \end{aligned}$$

Moreover, using Definition 3.1 and (3.4) we derive

$$g(\nabla_X Y, V) = g(A_X BY, JV) - g(\mathcal{H}\,\mathrm{grad}\ln\lambda, CY)g(X, JV) + g(\mathcal{H}\,\mathrm{grad}\ln\lambda, JV)g(X, CY) + \frac{1}{\lambda^2}g'(\nabla_{F_*X}F_*CY, F_*JV).$$

Using (2.2) we conclude that $M_{(\ker F_*)^\perp}$ is totally umbilical if and only if the equation (5.2) is satisfied. $\square$

However, in the sequel, we show that the notion of conformal anti-invariant submersion puts some restrictions on the total space for locally warped product manifold.

Theorem 5.4. *Let F be a conformal anti-invariant submersion from a Kähler manifold (M, g, J) to a Riemannian manifold (N, g') with rank$(\ker F_*) > 1$. If M is a locally warped product manifold of the form $M_{(\ker F_*)^\perp} \times_\lambda M_{(\ker F_*)}$, then either F is horizontally homothetic submersion or the fibers are one dimensional.*

Proof. For $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$, from (2.2) and (2.6) we get

$$-X(\ln\lambda)g(U, V) = JV(\ln\lambda)g(U, JX).$$

For $X \in \Gamma(\mu)$, we derive

$$-X(\ln\lambda)g(U, V) = 0.$$

From above equation, we conclude that λ is a constant on $\Gamma(\mu)$. For $X = JU \in \Gamma(J(\ker F_*))$ we obtain

$$JU(\ln\lambda)g(U, V) = JV(\ln\lambda)g(U, U). \tag{5.3}$$

Interchanging the roles of V and U in (5.3) we arrive at

$$JV(\ln\lambda)g(U, V) = JU(\ln\lambda)g(V, V). \tag{5.4}$$

From (5.3) and (5.4) we get

$$JU(\ln \lambda) = JU(\ln \lambda)\frac{g(U,V)^2}{\| U \|^2 \| V \|^2}. \tag{5.5}$$

From (5.5), either λ is a constant on $\Gamma(J \ker F_*)$ or $\Gamma(J \ker F_*)$ is 1-dimensional. Thus proof is complete. $\square$

Remark 5.5. *In fact, the result implies that there are no conformal anti-invariant submersions from Kähler manifold (M, g, J) the form $M_{(\ker F_*)^\perp} \times_\lambda M_{(\ker F_*)}$ to a Riemannian manifold under certain conditions.*

6. **Curvature Relations for Conformal Anti-Invariant Submersions**

In this section, we investigate sectional curvatures of the total space, the base space and the fibres of a conformal anti-invariant submersion. Let F be a conformal anti-invariant submersion between Kähler manifold M and Riemannian manifold N. We denote Riemannian curvature tensors of M, N and any fibre $F^{-1}(x)$ by R_M, R_N and $\hat{R}$, respectively.

Let F be a conformal anti-invariant submersion from a Kähler manifold (M, g, J) to a Riemannian manifold (N, g'). We denote by K the sectional curvature, defined for any pair of non zero orthogonal vectors X and Y on M by the formula:

$$K(X,Y) = \frac{R(X,Y,Y,X)}{\|X\|^2\|Y\|^2}. \tag{6.1}$$

We denote sectional curvatures of M, N and any fibre $F^{-1}(x)$ by K_M, K_N and $\hat{K}$ respectively.

Theorem 6.1. *Let F be a conformal anti-invariant submersion from a Kähler manifold (M, g, J) to a Riemannian manifold (N, g') and let $K_M, \hat{K}$ and K_N be the sectional curvatures of the total space M, fibers and the base space N, respectively. If X, Y, Z, H*

are horizontal and U,V,W,F *vertical vectors, then*

$$K_M(U,V)=\frac{1}{\lambda^2}K_N(JU,JV)-\frac{3}{4}\parallel\mathcal{V}\,[JU,JV]\parallel^2-\frac{\lambda^2}{2}\{g(\nabla_{JU}\operatorname{grad}(\frac{1}{\lambda^2}),JU)$$
$$+g(\nabla_{JV}\operatorname{grad}(\frac{1}{\lambda^2}),JV)\}+\frac{\lambda^4}{4}\{\parallel\operatorname{grad}(\frac{1}{\lambda^2})\parallel^2$$
$$+\parallel JU(\frac{1}{\lambda^2})JV-JV(\frac{1}{\lambda^2})JU\parallel^2\},\tag{6.2}$$

$$K_M(X,Y)=\hat{K}(BX,BY)+\frac{1}{\lambda^2}K_N(CX,CY)-\frac{3}{4}\parallel\mathcal{V}\,[CX,CY]\parallel^2$$
$$+\frac{\lambda^2}{2}\{g(CX,CY)g(\nabla_{CY}\operatorname{grad}(\frac{1}{\lambda^2}),CX)$$
$$-g(CY,CY)g(\nabla_{CX}\operatorname{grad}(\frac{1}{\lambda^2}),CX)+g(CY,CX)g(\nabla_{CX}\operatorname{grad}(\frac{1}{\lambda^2}),CY)$$
$$-g(CX,CX)g(\nabla_{CY}\operatorname{grad}(\frac{1}{\lambda^2}),CY)\}$$
$$+\frac{\lambda^4}{4}\{(g(CX,CX)g(CY,CY)-g(CY,CX)g(CX,CY))\parallel\operatorname{grad}(\frac{1}{\lambda^2})\parallel^2$$
$$+\parallel CX(\frac{1}{\lambda^2})CY-CY(\frac{1}{\lambda^2})CX\parallel^2\}+\parallel T_{BX}BX\parallel^2-g(T_{BY}BY,T_{BX}BX)$$
$$+g((\nabla_{BX}A)_{CY}CY,BX)+\parallel A_{CY}BX\parallel^2-g((\nabla_{CY}T)_{BX}CY,BX)$$
$$-\parallel T_{BX}CY\parallel^2+g((\nabla_{BY}A)_{CX}CX,BY)+\parallel A_{CX}BY\parallel^2$$
$$-g((\nabla_{CX}T)_{BY}CX,BY)-\parallel T_{BY}CX\parallel^2\tag{6.3}$$

and

$$K_M(X,U)=\frac{1}{\lambda^2}K_N(CX,JU)-\frac{3}{4}\parallel\mathcal{V}\,[CX,JU]\parallel^2$$
$$-\frac{\lambda^2}{2}\{g(CX,CX)g(\nabla_{JU}\operatorname{grad}(\frac{1}{\lambda^2}),JU)$$
$$+g(\nabla_{CX}\operatorname{grad}(\frac{1}{\lambda^2}),CX)\}+\frac{\lambda^4}{4}\{g(CX,CX)\parallel\operatorname{grad}(\frac{1}{\lambda^2})\parallel^2$$
$$+\parallel CX(\frac{1}{\lambda^2})JU-JU(\frac{1}{\lambda^2})CX\parallel^2\}+g((\nabla_{BX}A)_{JU}JU,BX)+\parallel A_{JU}BX\parallel^2$$
$$-g((\nabla_{JU}T)_{BX}JU,BX)-\parallel T_{BX}JU\parallel^2\,.\tag{6.4}$$

Proof. Since M is a Kähler manifold, we have $K_M(U,V) = K_M(JU,JV)$. Considering (2.11) and (6.1), we obtain

$$K_M(U,V) = K_M(JU,JV) = g(R_M(JU,JV)JV,JU) = \frac{1}{\lambda^2}g'(R_N(JU,JV)JV,JU)$$
$$+\frac{1}{4}\{g(\mathcal{V}[JU,JV],\mathcal{V}[JV,JU]) - g(\mathcal{V}[JV,JV],\mathcal{V}[JU,JU])$$
$$+2g(\mathcal{V}[JU,JV],\mathcal{V}[JV,JU])\}$$
$$+\frac{\lambda^2}{2}\{g(JU,JV)g(\nabla_{JV}\operatorname{grad}(\frac{1}{\lambda^2}),JU) - g(JV,JV)g(\nabla_{JU}\operatorname{grad}(\frac{1}{\lambda^2}),JU)$$
$$+g(JV,JU)g(\nabla_{JU}\operatorname{grad}(\frac{1}{\lambda^2}),JV) - g(JU,JU)g(\nabla_{JV}\operatorname{grad}(\frac{1}{\lambda^2}),JV)\}$$
$$+\frac{\lambda^4}{4}\{(g(JU,JU)g(JV,JV) - g(JV,JU)g(JU,JV)) \parallel \operatorname{grad}(\frac{1}{\lambda^2}) \parallel^2$$
$$+g(JU(\frac{1}{\lambda^2})JV - JV(\frac{1}{\lambda^2})JU, JU(\frac{1}{\lambda^2})JV - JV(\frac{1}{\lambda^2})JU)\}$$

for unit vector fields U and V. By straightforward computations, we get (6.1).

For unit vector fields X and Y, since M is a Kähler manifold and using (3.2), we have

$$K_M(X,Y) = K_M(JX,JY) = K_M(BX,BY) + K_M(CX,CY) \tag{6.5}$$
$$+K_M(BX,CY) + K_M(CX,BY).$$

Using (2.11), we derive

$$K_M(BX,BY) = g(R_M(BX,BY)BY,BX) = g(\hat{R}(BX,BY)BY,BX)$$
$$+g(T_{BX}BY,T_{BY}BX) - g(T_{BY}BY,T_{BX}BX)$$
$$= \hat{K}(BX,BY) + \parallel T_{BX}BY \parallel^2 - g(T_{BY}BY,T_{BX}BX). \tag{6.6}$$

In a similar way, using (2.14), we arrive at

$$K_M(CX,CY) = g(R_M(CX,CY)CY,CX) = \frac{1}{\lambda^2}g'(R_N(CX,CY)CY,CX)$$
$$+\frac{1}{4}\{g(\mathcal{V}[CX,CY],\mathcal{V}[CY,CX]) - g(\mathcal{V}[CY,CY],\mathcal{V}[CX,CX])$$

$$+2g(\mathcal{V}[CX,CY],\mathcal{V}[CY,CX])\}$$
$$+\frac{\lambda^2}{2}\{g(CX,CY)g(\nabla_{CY}\operatorname{grad}(\frac{1}{\lambda^2}),CX)-g(CY,CY)g(\nabla_{CX}\operatorname{grad}(\frac{1}{\lambda^2}),CX)$$
$$+g(CY,CX)g(\nabla_{CX}\operatorname{grad}(\frac{1}{\lambda^2}),CY)-g(CX,CX)g(\nabla_{CY}\operatorname{grad}(\frac{1}{\lambda^2}),CY)\}$$
$$+\frac{\lambda^4}{4}\{(g(CX,CX)g(CY,CY)-g(CY,CX)g(CX,CY))\parallel\operatorname{grad}(\frac{1}{\lambda^2})\parallel^2$$
$$+g(CX(\frac{1}{\lambda^2})CY-CY(\frac{1}{\lambda^2})CX,CX(\frac{1}{\lambda^2})CY-CY(\frac{1}{\lambda^2})CX)\}.$$

Also by direct calculations, we obtain

$$K_M(CX,CY)=\frac{1}{\lambda^2}K_N(CX,CY)-\frac{3}{4}\parallel\mathcal{V}[CX,CY]\parallel^2$$
$$+\frac{\lambda^2}{2}\{g(CX,CY)g(\nabla_{CY}\operatorname{grad}(\frac{1}{\lambda^2}),CX)-g(CY,CY)g(\nabla_{CX}\operatorname{grad}(\frac{1}{\lambda^2}),CX)$$
$$+g(CY,CX)g(\nabla_{CX}\operatorname{grad}(\frac{1}{\lambda^2}),CY)-g(CX,CX)g(\nabla_{CY}\operatorname{grad}(\frac{1}{\lambda^2}),CY)\}$$
$$+\frac{\lambda^4}{4}\{(g(CX,CX)g(CY,CY)-g(CY,CX)g(CX,CY))\parallel\operatorname{grad}(\frac{1}{\lambda^2})\parallel^2$$
$$+\parallel CX(\frac{1}{\lambda^2})CY-CY(\frac{1}{\lambda^2})CX,CX(\frac{1}{\lambda^2})CY\parallel^2\}.\tag{6.7}$$

In a similar way, using (2.1) we have

$$K_M(BX,CY)=g(R_M(BX,CY)CY,BX)=g((\nabla_{BX}A)_{CY}CY,BX)+\parallel A_{CY}BX\parallel^2$$
$$-g((\nabla_{CY}T)_{BX}CY,BX)-\parallel T_{BX}CY\parallel^2.\tag{6.8}$$

Lastly, since M is a Kähler manifold and using (2.13) we obtain

$$K_M(CX,BY)=K_M(BY,CX)=g(R_M(BY,CX)CX,BY)=g((\nabla_{BY}A)_{CX}CX,BY)$$
$$+\parallel A_{CX}BY\parallel^2-g((\nabla_{CX}T)_{BY}CX,BY)-\parallel T_{BY}CX\parallel^2.\tag{6.9}$$

Writing (6.6), (6.7), (6.8) and (6.9) in (6.5) we get (6.3).

For unit vector fields X and U, since M is a Kähler manifold and from (3.2), we have

$$K_M(X,U) = K_M(JX,JU) = K_M(BX,JU) + K_M(CX,JU). \tag{6.10}$$

Using (2.13), we get

$$K_M(BX,JU) = g(R_M(BX,JU)JU,BX) = g((\nabla_{BX}A)_{JU}JU,BX) + \| A_{JU}BX \|^2 \tag{6.11}$$
$$-g((\nabla_{JU}T)_{BX}JU,BX) - \| T_{BX}JU \|^2 .$$

In a similar way, using (2.14) we obtain

$$\begin{aligned}
&K_M(CX,JU) = g(R_M(CX,JU)JU,CX) = \frac{1}{\lambda^2}g'(R_N(CX,JU)JU,CX)\\
&+\frac{1}{4}\{g(\mathcal{V}[CX,JU],\mathcal{V}[JU,CX]) - g(\mathcal{V}[JU,JU],\mathcal{V}[CX,CX])\\
&+2g(\mathcal{V}[CX,JU],\mathcal{V}[JU,CX])\}\\
&+\frac{\lambda^2}{2}\{g(CX,JU)g(\nabla_{JU}\operatorname{grad}(\frac{1}{\lambda^2}),CX) - g(JU,JU)g(\nabla_{CX}\operatorname{grad}(\frac{1}{\lambda^2}),CX)\\
&+g(JU,CX)g(\nabla_{CX}\operatorname{grad}(\frac{1}{\lambda^2}),JU) - g(CX,CX)g(\nabla_{JU}\operatorname{grad}(\frac{1}{\lambda^2}),JU)\}\\
&+\frac{\lambda^4}{4}\{(g(CX,CX)g(JU,JU) - g(JU,CX)g(CX,JU)) \| \operatorname{grad}(\frac{1}{\lambda^2}) \|^2\\
&+ \| CX(\frac{1}{\lambda^2})JU - JU(\frac{1}{\lambda^2})CX \|^2\}.
\end{aligned} \tag{6.12}$$

If we write (6.11) and (6.12) in (6.10) and arranging the equation, we get (6.4). □

From Theorem 6.1, we have the following results.

Corollary 6.2. *Let F be a conformal anti-invariant submersion from a Kähler manifold (M,g,J) to a Riemannian manifold (N,g'). Then we have,*

$$\begin{aligned}
\hat{K}(U,V) \leq{}& \frac{1}{\lambda^2}K_N(JU,JV) - \frac{\lambda^2}{2}\{g(\nabla_{JU}\operatorname{grad}(\frac{1}{\lambda^2}),JU) + g(\nabla_{JV}\operatorname{grad}(\frac{1}{\lambda^2}),JV)\}\\
&+\frac{\lambda^4}{4}\{\| \operatorname{grad}(\frac{1}{\lambda^2}) \|^2 + \| JU(\frac{1}{\lambda^2})JV - JV(\frac{1}{\lambda^2})JU \|^2\} + g(T_VV,T_UU)
\end{aligned}$$

for $U,V\in\Gamma(\ker F_*)$. *The equality case is satisfied if and only if the fibers are totally geodesic and* $J\ker F_*$ *is integrable.*

Proof. From (6.2), we have

$$K_M(U,V)=\frac{1}{\lambda^2}K_N(JU,JV)-\frac{3}{4}\parallel\mathcal{V}\,[JU,JV]\parallel^2-\frac{\lambda^2}{2}\{g(\nabla_{JU}\operatorname{grad}(\frac{1}{\lambda^2}),JU)$$
$$+g(\nabla_{JV}\operatorname{grad}(\frac{1}{\lambda^2}),JV)\}+\frac{\lambda^4}{4}\{\parallel\operatorname{grad}(\frac{1}{\lambda^2})\parallel^2+\parallel JU(\frac{1}{\lambda^2})JV-JV(\frac{1}{\lambda^2})JU\parallel^2\}.$$

Using ([16], Corollary 1, page: 465), we get

$$\begin{aligned}\hat{K}(U,V)+\parallel T_UV\parallel^2-g(T_VV,T_UU)&=\frac{1}{\lambda^2}K_N(JU,JV)-\frac{3}{4}\parallel\mathcal{V}\,[JU,JV]\parallel^2\\&-\frac{\lambda^2}{2}\{g(\nabla_{JU}\operatorname{grad}(\frac{1}{\lambda^2}),JU)+g(\nabla_{JV}\operatorname{grad}(\frac{1}{\lambda^2}),JV)\}\\&+\frac{\lambda^4}{4}\{\parallel\operatorname{grad}(\frac{1}{\lambda^2})\parallel^2+\parallel JU(\frac{1}{\lambda^2})JV-JV(\frac{1}{\lambda^2})JU\parallel^2\}\end{aligned}\tag{6.13}$$

which gives the assertion. □

We also have the following result.

Corollary 6.3. *Let* F *be a conformal anti-invariant submersion from a Kähler manifold* (M,g,J) *to a Riemannian manifold* (N,g')*. Then we have,*

$$\hat{K}(U,V)\geq\frac{1}{\lambda^2}K_N(JU,JV)-\frac{3}{4}\parallel\mathcal{V}\,[JU,JV]\parallel^2-\frac{\lambda^2}{2}\{g(\nabla_{JU}\operatorname{grad}(\frac{1}{\lambda^2}),JU)$$
$$+g(\nabla_{JV}\operatorname{grad}(\frac{1}{\lambda^2}),JV)\}-\parallel T_UV\parallel^2+g(T_VV,T_UU)$$

for $U,V\in\Gamma(\ker F_*)$. *The equality case is satisfied if and only if* F *is a homotetic submersion.*

Corollary 6.4. *Let F be a conformal anti-invariant submersion from a Kähler manifold (M,g,J) to a Riemannian manifold (N,g'). Then we have,*

$$
\begin{aligned}
K_M(X,Y) \geq\; & \hat{K}(BX,BY) + \frac{1}{\lambda^2}K_N(CX,CY) - \frac{3}{4} \parallel \mathcal{V}[CX,CY] \parallel^2 \\
& + \frac{\lambda^2}{2}\{g(CX,CY)g(\nabla_{CY}\operatorname{grad}(\frac{1}{\lambda^2}),CX) \\
& - g(CY,CY)g(\nabla_{CX}\operatorname{grad}(\frac{1}{\lambda^2}),CX) + g(CY,CX)g(\nabla_{CX}\operatorname{grad}(\frac{1}{\lambda^2}),CY) \\
& - g(CX,CX)g(\nabla_{CY}\operatorname{grad}(\frac{1}{\lambda^2}),CY)\} \\
& + \frac{\lambda^4}{4}\{(g(CX,CX)g(CY,CY) - g(CY,CX)g(CX,CY)) \parallel \operatorname{grad}(\frac{1}{\lambda^2}) \parallel^2\} \\
& - g(T_{BY}BY, T_{BX}BX) - g((\nabla_{CY}T)_{BX}CY, BX) + g((\nabla_{BX}A)_{CY}CY, BX) \\
& - \parallel T_{BX}CY \parallel^2 + g((\nabla_{BY}A)_{CX}CX, BY) - g((\nabla_{CX}T)_{BY}CX, BY) - \parallel T_{BY}CX \parallel^2
\end{aligned}
$$

for $X,Y \in \Gamma((\ker F_*)^\perp)$. The equality case is satisfied if and only if $T_{BX}BX = 0$, $A_{CY}BX = 0$ and $CX(\frac{1}{\lambda^2})CY - CY(\frac{1}{\lambda^2})CX = 0$ which shows that either μ is one dimensional or λ is a constant on μ.

Proof. By direct calculations and using (6.3) we arrive at,

$$
\begin{aligned}
& K_M(X,Y) - \parallel T_{BX}BX \parallel^2 - \parallel A_{CY}BX \parallel^2 - \parallel CX(\frac{1}{\lambda^2})CY - CY(\frac{1}{\lambda^2})CX \parallel^2 \\
& = \hat{K}(BX,BY) + \frac{1}{\lambda^2}K_N(CX,CY) - \frac{3}{4} \parallel \mathcal{V}[CX,CY] \parallel^2 \\
& \frac{\lambda^2}{2}\{-g(CY,CY)g(\nabla_{CX}\operatorname{grad}(\frac{1}{\lambda^2}),CX) + g(CY,CX)g(\nabla_{CX}\operatorname{grad}(\frac{1}{\lambda^2}),CY) \\
& -g(CX,CX)g(\nabla_{CY}\operatorname{grad}(\frac{1}{\lambda^2}),CY) + g(CX,CY)g(\nabla_{CY}\operatorname{grad}(\frac{1}{\lambda^2}),CX)\} \\
& +\frac{\lambda^4}{4}\{(g(CX,CX)g(CY,CY) - g(CY,CX)g(CX,CY)) \parallel \operatorname{grad}(\frac{1}{\lambda^2}) \parallel^2\} \\
& -g(T_{BY}BY, T_{BX}BX) - g((\nabla_{CY}T)_{BX}CY, BX) + g((\nabla_{BX}A)_{CY}CY, BX) \\
& - \parallel T_{BX}CY \parallel^2 + g((\nabla_{BY}A)_{CX}CX, BY) - g((\nabla_{CX}T)_{BY}CX, BY) - \parallel T_{BY}CX \parallel^2 \\
& + \parallel A_{CX}BY \parallel^2 .
\end{aligned}
$$

This gives the inequality. For the equality case $\| T_{BX}BX \|^2 + \| A_{CY}BX \|^2 + \| CX(\frac{1}{\lambda^2})CY - CY(\frac{1}{\lambda^2})CX \|^2 = 0$. Hence we obtain $T_{BX}BX = 0$, $A_{CY}BX = 0$ and $CX(\frac{1}{\lambda^2})CY - CY(\frac{1}{\lambda^2})CX = 0$ which shows that either μ is one dimensional or λ is a constant on μ. $\square$

In a similar way, we have the following result.

Corollary 6.5. *Let F be a conformal anti-invariant submersion from a Kähler manifold (M, g, J) to a Riemannian manifold (N, g'). Then we have,*

$$\begin{aligned} K_M(X,Y) \leq \hat{K}(BX,BY) &+ \frac{1}{\lambda^2}K_N(CX,CY) - \frac{\lambda^2}{2}\{g(CX,CY)g(\nabla_{CY}\operatorname{grad}(\frac{1}{\lambda^2}),CX) \\ &- g(CY,CY)g(\nabla_{CX}\operatorname{grad}(\frac{1}{\lambda^2}),CX) + g(CY,CX)g(\nabla_{CX}\operatorname{grad}(\frac{1}{\lambda^2}),CY) \\ &- g(CX,CX)g(\nabla_{CY}\operatorname{grad}(\frac{1}{\lambda^2}),CY)\} \\ &+ \frac{\lambda^4}{4}\{(g(CX,CX)g(CY,CY) - g(CY,CX)g(CX,CY)) \| \operatorname{grad}(\frac{1}{\lambda^2}) \|^2 \\ &+ \| CX(\frac{1}{\lambda^2})CY - CY(\frac{1}{\lambda^2})CX \|^2\} + \| T_{BX}BX \|^2 - g(T_{BY}BY, T_{BX}BX) \\ &+ g((\nabla_{BX}A)_{CY}CY,BX) + \| A_{CY}BX \|^2 - g((\nabla_{CY}T)_{BX}CY,BX) \\ &+ g((\nabla_{BY}A)_{CX}CX,BY) + \| A_{CX}BY \|^2 - g((\nabla_{CX}T)_{BY}CX,BY) - \| T_{BY}CX \|^2 \end{aligned}$$

for $X, Y \in \Gamma((\ker F_*)^{\perp})$. The equality case is satisfied if and only if $T_{BX}CY = 0$ and $[CX, CY] \in \Gamma(\mathcal{H})$.

Corollary 6.6. *Let F be a conformal anti-invariant submersion from a Kähler manifold (M, g, J) to a Riemannian manifold (N, g'). Then we have,*

$$\begin{aligned} K_M(X,U) \geq \frac{1}{\lambda^2}&K_N(CX,JU) - \frac{3}{4} \| \mathcal{V}[CX,JU] \|^2 \\ &- \frac{\lambda^2}{2}\{g(CX,CX)g(\nabla_{JU}\operatorname{grad}(\frac{1}{\lambda^2}),JU) \\ &+ g(\nabla_{CX}\operatorname{grad}(\frac{1}{\lambda^2}),CX)\} + g((\nabla_{BX}A)_{JU}JU,BX) \end{aligned}$$

$$- g((\nabla_{JU}T)_{BX}JU, BX) - \| T_{BX}JU \|^2$$

for $X \in \Gamma((\ker F_*)^\perp)$ *and* $U \in \Gamma(\ker F_*)$. *The equality case is satisfied if and only if* $A_{JU}BX = 0$, $\mathrm{grad}(\frac{1}{\lambda^2}) = 0$ *and* F *horizontally homothetic submersion.*

Proof. By straightforward computations and using (6.4) we obtain,

$$\begin{aligned}
&K_M(X,U) - \| A_{JU}BX \|^2 - \frac{\lambda^4}{4}\{g(CX,CX) \| \mathrm{grad}(\frac{1}{\lambda^2}) \|^2 \\
&+ \| CX(\frac{1}{\lambda^2})JU - JU(\frac{1}{\lambda^2})CX \|^2\} = \frac{1}{\lambda^2}K_N(CX,JU) - \frac{3}{4} \| \mathcal{V}[CX,JU] \|^2 \\
&-\frac{\lambda^2}{2}\{g(CX,CX)g(\nabla_{JU}\mathrm{grad}(\frac{1}{\lambda^2}),JU) + g(\nabla_{CX}\mathrm{grad}(\frac{1}{\lambda^2}),CX)\} \\
&+g((\nabla_{BX}A)_{JU}JU,BX) - g((\nabla_{JU}T)_{BX}JU,BX) - \| T_{BX}JU \|^2 .
\end{aligned}$$

This gives the inequality. For the equality case $\| A_{JU}BX \|^2 + \frac{\lambda^4}{4}\{g(CX,CX) \| \mathrm{grad}(\frac{1}{\lambda^2}) \|^2 + \| CX(\frac{1}{\lambda^2})JU - JU(\frac{1}{\lambda^2})CX \|^2\} = 0$. Thus we derive $A_{JU}BX = 0$ and $\mathrm{grad}(\frac{1}{\lambda^2}) = 0$, $CX(\frac{1}{\lambda^2})JU - JU(\frac{1}{\lambda^2})CX = 0$ which shows that F is horizontally homotetic. □

Finally we have the following inequality.

Corollary 6.7. *Let* F *be a conformal anti-invariant submersion from a Kähler manifold* (M,g,J) *to a Riemannian manifold* (N,g'). *Then we have,*

$$\begin{aligned}
K_M(X,U) \leq \frac{1}{\lambda^2}K_N(CX,JU) &- \frac{\lambda^2}{2}\{g(CX,CX)g(\nabla_{JU}\mathrm{grad}(\frac{1}{\lambda^2}),JU) \\
&+ g(\nabla_{CX}\mathrm{grad}(\frac{1}{\lambda^2}),CX)\} \\
&+ \frac{\lambda^4}{4}\{g(CX,CX) \| \mathrm{grad}(\frac{1}{\lambda^2}) \|^2 + \| CX(\frac{1}{\lambda^2})JU - JU(\frac{1}{\lambda^2})CX \|^2\} \\
&+ g((\nabla_{BX}A)_{JU}JU,BX) + \| A_{JU}BX \|^2 - g((\nabla_{JU}T)_{BX}JU,BX)
\end{aligned}$$

for $X \in \Gamma((\ker F_*)^\perp)$ *and* $U \in \Gamma(\ker F_*)$. *The equality case is satisfied if and only if* $T_{BX}JU = 0$ *and* $[CX,JU] \in \mathcal{H}$.

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